Global Stability of Worm Propagation Model with Nonlinear Incidence Rate in Computer Network

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Abstract

In this paper, an e-epidemic $SVEIS$ model describing the transmission of worms with nonlinear incidence rate through horizontal transmission is formulated in computer network. The existence of two equilibrium points: worm-free and endemic equilibria have been investigated. The stability analyses are determined by the basic reproduction number. It has been observed that if the basic reproduction number,

$$R_0 = \frac{\sigma\beta(\mu+\theta)\Lambda}{\eta_2(\mu+\gamma)(\Lambda(\mu+\theta)+\eta_1\mu c)} < 1,$$

the system is globally asymptotically stable and the infected nodes get vanish at worm-free equilibrium state; worms fade out from the network. However, if $R_0 > 1$, the infected node exists; worm persists in the network at an endemic equilibrium state and is globally stable with some conditions. Further, the transcritical bifurcation at $R_0 = 1$, has obtained using the center manifold theorem. The effect of vaccination and non-linear transmission rate on the dynamics of the model system has been observed. The dynamical behavior of the susceptible, exposed and infected nodes with real parametric values is examined. We also observe that the critical vaccination rate is required to eradicate the worm. Our results illustrate several administrative and executive insights.

Keywords: Computer Network; E-Epidemic Model; Nonlinear Incidence Rate; Stability Analysis; Transcritical Bifurcation; Worm Propagation

1 Introduction

Over the last several decades, a rigorous global effort is speeding up the developments, in the establishment of a worldwide surveillance network for the propagation of computer malicious objects (Viruses, Worms and Trojans). Researchers from computer science and applied mathematics have collaborated for fast assessment of potentially critical conditions. To achieve this goal, mathematical modeling plays a vital role in efforts; that focuses on predicting, assessing and controlling potential outbreak. Also, the epidemic modeling and its applications have been used to understand the effect of changes in the behavior of solutions of the model system. To better understand such dynamics, the papers of Kermack and McKendrick [13] and Capasso and Serio [2] can be studied which established the deterministic compartmental epidemic modeling. In this way, many research articles have published and discussed the main approaches that are used for the surveillance and modeling of biological diseases as well as computer viruses dynamics. Wang *et al.* [26] proposed a novel worm attack $SVEIR$ model using saturated incidence rate and partial immunization rate. In which they have shown the partial immunization is highly effective for eliminating worms. A propagation model with varying node numbers of removable memory device(RMD) virus have been formulated and obtained three threshold parameters to control the RMD-virus in [11].

Worm exploits security vulnerabilities and does not require any user action to propagate. It is a self-propagating malicious program that focuses mainly on infecting as many nodes as possible to the network. Several network phenomena are well modeled as transmissions (through both horizontally and vertically) of viruses or worms through a network. We consider the vaccination strategies that are used to control the spreading of malicious objects [24, 30]. The regular pattern of periodic occurrences have been observed in the epidemiology of many infectious diseases and computer viruses. To predict and control the spread of computer worms, it is necessary to understand such periodic patterns and identify the specific factors that exhibit such periodic outbreak [16]. Zhang *et al.* [29] have employed an impulsive state feedback model to study the transmission of computer worm and the preventive effect of operating system patching.

Recent studies have demonstrated that the nonlinear incidence rate is one of the important factor for the modeling of epidemic and e-epidemic systems that induces periodic oscillations in epidemic models [23, 27]. For modeling the transmission process, researchers have employed different forms of incidence rate on which the dynamics of the model system depends extensively. The e-epidemicity of worm is closely related to the stability of the equilibria of the model system. Many researchers have considered bilinear incidence rate (βSI) [7, 14], standard incidence rate $\frac{\beta SI}{N}$ [18], nonlinear incidence rate $\frac{\beta SI}{f(I)}$ [8], modified saturated incidence rate $f(S,I) = \frac{\beta SI}{(1+\alpha S+\gamma I)}$(Beddington-DeAngelis type), where $\alpha, \beta, \gamma > 0$ [5, 12], *etc.* in their studies. The use of classical e-epidemic transmission for studying computer virus propagation has been investigated by Piqueira *et al.* [21]. In the present work, we have taken nonlinear incidence rate as $f(S,I) = \frac{\beta SI}{(S+I+c)}$ [25].

In the e-epidemiology of worm propagation in the computer network, Mishra and Pandey [19, 18] described the effect of anti-virus software and vaccination on the attack of computer worms with global stability. Also, some research articles appear on the computer worm or virus model with different recovery rates and dynamics [22, 28]. Recently, Upadhyay *et al.* [25] proposed a $SVEIR$ model with nonlinear incidence rate for modeling the virus dynamics in computer network. In this paper, $SVEIS$ model with nonlinear incidence rate and vaccination strength are presented. This work is basically for the implementation and practice to predict and minimize the severe attack of worms in the computer network.

Here, we have investigated the global stability of the proposed e-epidemic $SVEIS$ model using modified nonlinear incidence rate and predict its optimal vaccination and eradicate worms from the network. The paper is structured as follows. In Section 2, we formulate an e-epidemic $SVEIS$ model as the system of ordinary differential equations and give the descriptions of all the parameters used in the model system. We find the two possible equilibrium points and its existence criteria and also calculate the basic reproduction number in Section 3. The stability analysis for both the equilibrium points are analyzed and transcritical bifurcation analysis is executed when basic reproduction number $R_0 = 1$ in Section 4. Section 5 presents the numerical simulations to verify the results found analytically by taking computer relevant value of parameters and discusses the stability of the model system using MATLAB and Mathematica. Finally, we conclude this article in Section 6.

2 Formulation of the Mathematical Model

Consider N nodes which have been divided into four subclasses as susceptible (S), vaccinated (V), exposed (E) and infectious (I) nodes and $N = S+V+E+I$. Some assumptions for formulating the model system are as follows:

1) We assume that any new node entering into the network is susceptible. The crashing rate of a node (due to hardware or software problems) μ is constant throughout the network.

2) The nodes are interacting heterogeneously. Worms are transmitted to the node through horizontal transmission.

3) The worms propagate into network when an infected file is transferred from an infectious node to the susceptible node. We have considered the modified nonlinear incidence rate $f(S,I) = \frac{\beta SI}{S+I+c}$. This represents the fact that the number of nodes carrying the worms can interact with other nodes, reaches some finite maximum value due to limitation of time or the network slowdown problems of the particular nodes [25].

4) Software offers temporary immunity to the nodes that is, when the software loss their efficiency or removed from the node of the computer network, the node becomes susceptible to attack again.

5) The worm induces temporary immunity is a fraction of recovered node, remaining recovered nodes again become susceptible [1]. A small fraction of exposed node recovers, rather than being infected and develops worm acquired temporary immunity due to self-prevention and detection techniques of operating system and becomes vaccinated [24]. A fraction of infected node after recovery gains temporary immunity against the worm and joins vaccinated class, remaining node becomes re-susceptible.

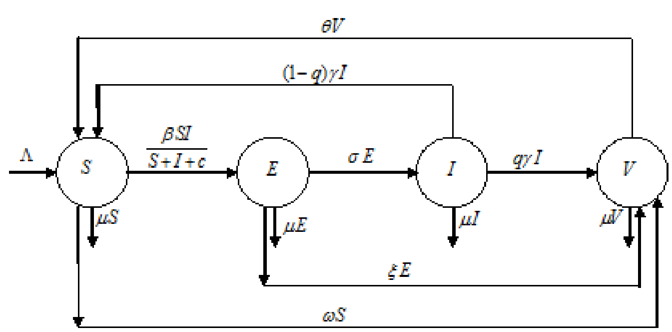

Figure 1: Schematic diagram for model system (1)

The schematic diagram for the model (1) is shown in Figure 1. The worm transmission between the different classes can be expressed by the following model:

$$\begin{cases} \frac{dS}{dt} &= \Lambda - \mu S - \omega S + \theta V - \frac{\beta SI}{S+I+c} + (1-q)\gamma I, \\ \frac{dV}{dt} &= \omega S - \theta V - \mu V + \xi E + q\gamma I, \\ \frac{dE}{dt} &= \frac{\beta SI}{S+I+c} - \mu E - \xi E - \sigma E, \\ \frac{dI}{dt} &= \sigma E - \mu I - \gamma I. \end{cases} \tag{1}$$

Table 1: Definition of parameters

Parameter	Descriptions	Units
S	Susceptible node	In number
V	Vaccinated node	”
E	Exposed node but not yet infectious	”
I	Infectious node	”
Λ	Rate at which new nodes are connected to the network	Day^{-1}
μ	Crashing rate of node due to hardware or software problems	”
ω	Vaccination rate	”
θ	Rate at which vaccinated nodes lose their immunity and join susceptible class	”
β	Contact rate or rate of transfer of worms from an infectious node to susceptible node	”
c	Half saturation constant	In number
q	Fraction of recovered nodes gaining worm-acquired immunity	Day^{-1}
ξ	Recovery rate of exposed class due to self-prevention technique of operating system	”
γ	Duration of infection of infected nodes	”
σ	Rate at which exposed node become infectious	”

with initial conditions: $S(0) = S_0 > 0, V(0) = V_0 > 0, E(0) = E_0 > 0, I(0) = I_0 > 0$, where all the parameters are positive and $0 \leq q \leq 1$. The definitions of all parameters are summarized in Table 1.

3 Existence of Equilibrium Points and Basic Reproduction Number

The existence of worm-free and endemic equilibrium points are established and basic reproduction number has been calculated.

We observe that total number of nodes N satisfies the equation $\frac{dN}{dt} = \Lambda - \mu N$ and hence $N(t) \to \frac{\Lambda}{\mu}$, as $t \to \infty$. The solutions of the model system (1) are non-negative for all $t \geq 0$. Therefore, the feasible region

$$U = \left\{(S, V, E, I) : 0 \leq S, V, E, I, N \leq \frac{\Lambda}{\mu}\right\},$$

is positively invariant in which the usual existence, uniqueness of solutions and continuation results hold.

The model system (1) always has the worm-free equilibrium $P^0 = \left(S^0, V^0, 0, 0\right)$, where

$$S^0 = \left(\frac{\mu+\theta}{\eta_1}\right) N^0, V^0 = \left(\frac{\omega}{\eta_1}\right) N^0, N^0 = \frac{\Lambda}{\mu},$$

with $\eta_1 = \mu + \theta + \omega$, represent the level of susceptible, vaccinated and total number of nodes respectively, in the absence of infection.

Now, we calculate the basic reproduction number [4]. Let $x = (E, I)$, then from the model system (1), it follows:

$$\frac{dx}{dt} = f - \upsilon,$$

where $f = \begin{bmatrix} \frac{\beta SI}{S+I+c} \\ 0 \end{bmatrix}$ and $\upsilon = \begin{bmatrix} \eta_2 E \\ -\sigma E + (\mu+\gamma) I \end{bmatrix}$.

We obtain $F =$ Jacobian of f at $WFE = \begin{bmatrix} 0 & \frac{\beta S^0}{S^0+c} \\ 0 & 0 \end{bmatrix}$ and $M =$ Jacobian of υ at $WFE = \begin{bmatrix} \eta_2 & 0 \\ -\sigma & \mu+\gamma \end{bmatrix}$.

The next generation matrix approach is used to compute the basic reproduction number, R_0 and is defined as the spectral radius of the next generation operator. The formation of the operator involves determining two compartments, infected and non-infected nodes for the considered model system.

The next generation matrix for the system is

$$K = FM^{-1} = \begin{bmatrix} \frac{\sigma\beta S^0}{\eta_2(\mu+\gamma)(S^0+c)} & \frac{\beta S^0}{(\mu+\gamma)(S^0+c)} \\ 0 & 0 \end{bmatrix}.$$

Thus, the basic reproduction number $R_0 = \rho(FM^{-1})$, of the model system (1) is given by

$$\begin{aligned} R_0 &= \frac{\sigma\beta S^0}{\eta_2(\mu+\gamma)(S^0+c)} \\ &= \frac{\sigma\beta(\mu+\theta)\Lambda}{\eta_2(\mu+\gamma)(\Lambda(\mu+\theta)+\eta_1\mu c)}. \end{aligned}$$

Further, the model system (1) also has an interior equilibrium given by $P^* = (S^*, V^*, E^*, I^*)$, where

$$\begin{aligned} S^* &= \frac{I^*+c}{(R_0-1)+c\frac{R_0}{S^0}}, E^* = \frac{\mu+\gamma}{\sigma} I^*, \\ V^* &= \frac{1}{\mu+\theta}\begin{bmatrix} \left(\frac{\omega}{(R_0-1)+c\frac{R_0}{S^0}} + \frac{\xi(\mu+\gamma)}{\sigma} + q\gamma\right) I^* \\ +\frac{\omega c}{(R_0-1)+c\frac{R_0}{S^0}} \end{bmatrix}, \\ I^* &= \frac{\Lambda\left(R_0 - 1 + c\frac{R_0}{S^0}\right)(\theta+\mu)\sigma - \eta_1 c\mu\sigma}{\mu\left(\left(R_0 - 1 + c\frac{R_0}{S^0}\right)\left\{\begin{array}{c}(\gamma+\mu)(\theta+\mu+\xi) \\ +(q\gamma+\theta+\mu)\sigma\end{array}\right\} + \sigma\eta_1\right)}. \end{aligned}$$

We conclude from the above that the endemic equilibrium point exists if $R_0 > 1$.

4 Stability Analysis of the Model System

We investigate the stability (linear as well as nonlinear) analysis of both the equilibrium points. The reduced limiting dynamical system is given by ([23])

$$\begin{cases} \frac{dS}{dt} = \frac{\Lambda}{\mu}(\mu+\theta) - \eta_1 S - \frac{\beta SI}{S+I+c} - \theta E - (\theta - p\gamma)I, \\ \frac{dE}{dt} = \frac{\beta SI}{S+I+c} - \eta_2 E, \\ \frac{dI}{dt} = \sigma E - (\mu+\gamma)I, \end{cases} \quad (2)$$

with initial conditions: $S(0) = S_0 > 0, E(0) = E_0 > 0, I(0) = I_0 > 0$. All the parameters are positive. Also $\eta_1 = (\mu + \omega + \theta)$, $\eta_2 = (\mu + \xi + \sigma)$ and $p = 1 - q$.

Now, the local stability for worm-free equilibrium (WFE) point is established as follows:

Theorem 1. *The WFE point P^0 is*

1) *Locally asymptotically stable, if $R_0 < 1$,*

2) *Unstable, if $R_0 > 1$ and*

3) *A transcritical bifurcation occurs at $R_0 = 1$.*

Proof. The Jacobian matrix J_0 at WFE is given by

$$J_0 = \begin{bmatrix} -\eta_1 & -\theta & -\frac{\beta S^0}{S^0+c} - \theta + p\gamma \\ 0 & -\eta_2 & \frac{\beta S^0}{S^0+c} \\ 0 & \sigma & -(\mu+\gamma) \end{bmatrix}.$$

The characteristic equation of J_0 is given by

$$(\lambda + \eta_1)[\lambda^2 + (\mu + \eta_2 + \gamma)\lambda + \eta_2(\mu+\gamma)(1 - R_0)] = 0.$$

One eigenvalue is clearly negative; remaining two eigenvalues depends on the sign of the basic reproduction number. If $R_0 < 1$, then remaining two eigenvalues of J_0 have negative real parts and if $R_0 > 1$, then one eigenvalue of J_0 has negative real part and other has positive real part. Hence, WFE is locally asymptotically stable, if $R_0 < 1$ and unstable, if $R_0 > 1$. Now, if $R_0 = 1$, then two eigenvalues of J_0 have negative real parts and one eigenvalue is zero.

Let $x = S - S^0, y = E, z = I$. Then, system (2) reduces to

$$\begin{cases} \frac{dx}{dt} = -Ax - \frac{B(1+r)(x+S^0)z}{x+z+c+S^0} - \theta y - \theta z + (1-q)\gamma z, \\ \frac{dy}{dt} = \frac{B(1+r)(x+S^0)z}{x+z+c+S^0} - (\mu+\xi+\sigma)y, \\ \frac{dz}{dt} = \sigma y - (\mu+\gamma)z, \end{cases} \quad (3)$$

where

$$A = \eta_1, B = \frac{\eta_2(\gamma+\mu)(S^0+c)}{\sigma S^0}, R_0 = 1 + r.$$

For showing the occurrence of transcritical bifurcation at $(R_0, (S, E, I)) = (1, (S^0, 0, 0))$, we write β in terms of R_0 and other parameters. Linearizing system (3) about equilibrium point $(r, (x, y, z)) = (0, (0, 0, 0))$. we obtain the Jacobian matrix

$$\begin{bmatrix} -A & -\theta & (1-q)\gamma - \theta - \frac{BS^0}{S^0+c} \\ 0 & -\eta_2 & \frac{BS^0}{S^0+c} \\ 0 & \sigma & -\gamma - \mu \end{bmatrix}. \quad (4)$$

The proof is done by projecting the flow onto the extended center manifold [9]. The eigen-vectors corresponding to the eigenvalues $\lambda_1 = 0, \lambda_2 = -A$ and $\lambda_3 = -\gamma - \mu - \eta_2$ when $R_0 = 1 (r = 0)$ are

$$e_1 = \begin{bmatrix} -a_1 \\ a_3 \\ 1 \end{bmatrix}, e_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, e_3 = \begin{bmatrix} a_2 \\ -a_4 \\ 1 \end{bmatrix}$$

respectively, where

$$\begin{aligned} a_1 &= \frac{(\gamma+\mu)(\theta+\mu+\xi) + (q\gamma+\theta+\mu)\sigma}{A\sigma}, \\ a_2 &= \frac{(\gamma-\theta+\mu)(\mu+\xi) + (q\gamma+\mu)\sigma}{\sigma(-A+\gamma+\mu+\eta_2)}, \\ a_3 &= \frac{\gamma+\mu}{\sigma} \text{ and } a_4 = \frac{\eta_2}{\sigma}. \end{aligned}$$

The model matrix P with its column vector as the eigenvector is

$$P = \begin{bmatrix} -a_1 & 1 & a_2 \\ a_3 & 0 & -a_4 \\ 1 & 0 & 1 \end{bmatrix},$$

and hence

$$P^{-1} = \frac{1}{a_3 + a_4} \begin{bmatrix} 0 & 1 & a_4 \\ a_3 + a_4 & a_1 + a_2 & -a_2 a_3 + a_1 a_4 \\ 0 & -1 & a_3 \end{bmatrix}.$$

Now, we have to find the nature of stability $(x, y, z) = (0, 0, 0)$ for r near zero. We obtain the transformation using the eigen basis $\{e_1, e_2, e_3\}$,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = P \begin{bmatrix} u \\ v \\ w \end{bmatrix} \text{ with inverse } \begin{bmatrix} u \\ v \\ w \end{bmatrix} = P^{-1} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

which transform system (3) into

$$\left. \begin{aligned} \begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -A & 0 \\ 0 & 0 & -\gamma-\mu-\eta_2 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} \\ + \begin{bmatrix} f(u,v,w,r) \\ g_1(u,v,w,r) \\ g_2(u,v,w,r) \end{bmatrix} \end{aligned} \right\} \quad (5)$$

$$\dot{r} = 0. \quad (6)$$

Here

$$\begin{aligned} f(u,v,w,r) = {} & l_1 u^2 + l_2 w^2 + l_3 uv + l_4 uw + l_5 ur + l_6 vw \\ & + l_7 wr + l_8 u^3 + l_9 w^3 + l_{10} uvw + l_{11} uw^2 \\ & + l_{12} u^2 v + l_{13} u^2 w + l_{14} u^2 r + l_{15} vw^2 \\ & + l_{16} w^2 r + l_{17} wv^2 + l_{18} uvr + l_{19} vwr \\ & + l_{20} uwr + l_{21} uv^2, \end{aligned}$$

$$
\begin{aligned}
g_1(u,v,w,r) &= m_2u^2+m_3w^2+m_4uv+m_5uw \\
&\quad +m_6ur+m_7vw+m_8wr+m_9u^3 \\
&\quad +m_{10}w^3+m_{11}uvw+m_{12}uw^2 \\
&\quad +m_{13}u^2v+m_{14}u^2w+m_{15}u^2r \\
&\quad +m_{16}vw^2+m_{17}w^2r+m_{18}wv^2 \\
&\quad +m_{19}uvr+m_{20}vwr+m_{21}uw \\
&\quad +m_{22}uv^2, \\
g_2(u,v,w,r) &= n_2u^2+n_3w^2+n_4uv+n_5uw+n_6ur \\
&\quad +n_7vw+n_8wr+n_9u^3+n_{10}w^3 \\
&\quad +n_{11}uvw+n_{12}uw^2+n_{13}u^2v+n_{14}u^2w \\
&\quad +n_{15}u^2r+n_{16}vw^2+n_{17}w^2r+n_{18}wv^2 \\
&\quad +n_{19}uvr+n_{20}vwr+n_{21}uwr+n_{22}uv^2.
\end{aligned}
$$

Where

$$
\begin{aligned}
l_1 &= -\frac{B(S^0+ca_1)}{(c+S^0)^2(a_3+a_4)}, \\
l_2 &= \frac{B(-S^0+ca_2)}{(c+S^0)^2(a_3+a_4)}, \\
l_3 &= l_6=\frac{Bc}{(c+S^0)^2(a_3+a_4)}, \\
l_4 &= -\frac{B(2S^0+ca_1-ca_2)}{(c+S^0)^2(a_3+a_4)}, \\
l_5 &= l_7=\frac{BS^0}{(a_3+a_4)(c+S^0)}, \\
l_8 &= -\frac{B(-1+a_1)(S^0+ca_1)}{(c+S^0)^3(a_3+a_4)}, \\
l_9 &= \frac{B(1+a_2)(S^0-ca_2)}{(c+S^0)^3(a_3+a_4)}, \\
l_{10} &= \frac{2B(-c+S^0+ca_1-ca_2)}{(c+S^0)^3(a_3+a_4)}, \\
l_{11} &= \frac{B(3S^0-a_2(2c-2S^0+ca_2)+a_1(c-S^0+2ca_2))}{(c+S^0)^3(a_3+a_4)}, \\
l_{13} &= \frac{B(3S^0-ca_1^2+(-c+S^0)a_2+2a_1(c-S^0+ca_2))}{(c+S^0)^3(a_3+a_4)}, \\
l_{12} &= \frac{B(-c+S^0+2ca_1)}{(c+S^0)^3(a_3+a_4)}, \\
l_{14} &= -\frac{B(S^0+ca_1)}{(c+S^0)^2(a_3+a_4)}, \\
l_{15} &= \frac{B(-c+S^0-2ca_2)}{(c+S^0)^3(a_3+a_4)}, \\
l_{16} &= \frac{B(-S^0+ca_2)}{(c+S^0)^2(a_3+a_4)}, \\
l_{17} &= l_{21}=-\frac{Bc}{(c+S^0)^3(a_3+a_4)}, \\
l_{18} &= l_{19}=\frac{Bc}{(c+S^0)^2(a_3+a_4)}, \\
l_{20} &= -\frac{B(2S^0+ca_1-ca_2)}{(c+S^0)^2(a_3+a_4)},
\end{aligned}
$$

$$
\begin{aligned}
m_1 &= -A, \\
m_2 &= -\frac{B(S^0+ca_1)(a_1+a_2-a_3-a_4)}{(c+S^0)^2(a_3+a_4)}, \\
m_3 &= \frac{B(-S^0+ca_2)(a_1+a_2-a_3-a_4)}{(c+S^0)^2(a_3+a_4)}, \\
m_4 &= m_7=\frac{Bc(a_1+a_2-a_3-a_4)}{(c+S^0)^2(a_3+a_4)}, \\
m_5 &= -\frac{B(2S^0+ca_1-ca_2)(a_1+a_2-a_3-a_4)}{(c+S^0)^2(a_3+a_4)}, \\
m_6 &= m_8=\frac{BS^0(a_1+a_2-a_3-a_4)}{(c+S^0)(a_3+a_4)}, \\
m_9 &= \frac{B(1-a_1)(S^0+ca_1)(a_1+a_2-a_3-a_4)}{(c+S^0)^3(a_3+a_4)}, \\
m_{10} &= \frac{B(1+a_2)(S^0-ca_2)(a_1+a_2-a_3-a_4)}{(c+S^0)^3(a_3+a_4)}, \\
m_{11} &= \frac{2B(-c+S^0+ca_1-ca_2)(a_1+a_2-a_3-a_4)}{(c+S^0)^3(a_3+a_4)}, \\
m_{12} &= \frac{\begin{pmatrix} B(3S^0-a_2(2c-2S^0+ca_2) \\ +a_1(c-S^0+2ca_2))(a_1+a_2-a_3-a_4) \end{pmatrix}}{(c+S^0)^3(a_3+a_4)}, \\
m_{13} &= \frac{B(-c+S^0+2ca_1)(a_1+a_2-a_3-a_4)}{(c+S^0)^3(a_3+a_4)}, \\
m_{14} &= -\frac{\begin{pmatrix} B(-3S^0+ca_1^2+(c-S^0)a_2 \\ -2a_1(c-S^0+ca_2))(a_1+a_2-a_3-a_4) \end{pmatrix}}{(c+S^0)^3(a_3+a_4)}, \\
m_{15} &= -\frac{B(S^0+ca_1)(a_1+a_2-a_3-a_4)}{(c+S^0)^2(a_3+a_4)}, \\
m_{16} &= -\frac{B(c-S^0+2ca_2)(a_1+a_2-a_3-a_4)}{(c+S^0)^3(a_3+a_4)}, \\
m_{17} &= \frac{B(-S^0+ca_2)(a_1+a_2-a_3-a_4)}{(c+S^0)^2(a_3+a_4)}, \\
m_{18} &= m_{22}=-\frac{Bc(a_1+a_2-a_3-a_4)}{(c+S^0)^3(a_3+a_4)}, \\
m_{19} &= m_{20}=m_{21}=\frac{Bc(a_1+a_2-a_3-a_4)}{(c+S^0)^2(a_3+a_4)}, \\
n_1 &= -\gamma-2\mu-\xi-\sigma, \\
n_2 &= \frac{B(S^0+ca_1)}{(c+S^0)^2(a_3+a_4)}, \\
n_3 &= \frac{B(S^0-ca_2)}{(c+S^0)^2(a_3+a_4)}, \\
n_4 &= n_7=-\frac{Bc}{(c+S^0)^2(a_3+a_4)}, \\
n_5 &= \frac{B(2S^0+ca_1-ca_2)}{(c+S^0)^2(a_3+a_4)}, \\
n_6 &= n_8=-\frac{BS^0}{(c+S^0)(a_3+a_4)}, \\
n_9 &= \frac{B(-1+a_1)(S^0+ca_1)}{(c+S^0)^3(a_3+a_4)},
\end{aligned}
$$

$$
\begin{aligned}
n_{10} &= \frac{B(1+a_2)(-S^0+ca_2)}{(c+S^0)^3(a_3+a_4)},\\
n_{11} &= -\frac{2B(-c+S^0+ca_1-ca_2)}{(c+S^0)^3(a_3+a_4)},\\
n_{12} &= \frac{\begin{pmatrix} B(-3S^0+a_1(-c+S^0-2ca_2) \\ +a_2(2c-2S^0+ca_2)) \end{pmatrix}}{(c+S^0)^3(a_3+a_4)},\\
n_{13} &= -\frac{B(-c+S^0+2ca_1)}{(c+S^0)^3(a_3+a_4)},\\
n_{14} &= \frac{B(-3S^0+ca_1^2+(c-S^0)a_2-2a_1(c-S^0+ca_2))}{(c+S^0)^3(a_3+a_4)},\\
n_{15} &= \frac{B(S^0+ca_1)}{(c+S^0)^2(a_3+a_4)},\\
n_{16} &= \frac{B(c-S^0+2ca_2)}{(c+S^0)^3(a_3+a_4)},\\
n_{17} &= \frac{B(S^0-ca_2)}{(c+S^0)^2(a_3+a_4)},\\
n_{18} &= n_{22} = \frac{Bc}{(c+S^0)^3(a_3+a_4)},\\
n_{19} &= n_{20} = -\frac{Bc}{(c+S^0)^2(a_3+a_4)},\\
n_{21} &= \frac{B(2S^0+ca_1-ca_2)}{(c+S^0)^2(a_3+a_4)}.
\end{aligned}
$$

Let

$$g(u,v,w,r) = \begin{bmatrix} g_1(u,v,w,r) \\ g_2(u,v,w,r) \end{bmatrix}.$$

We have from the existence theorem for center manifolds

$$W^c(0) = \left\{ \begin{array}{l} (u,v,w,r) \in \mathbb{R}^4 \middle| v = h_1(u,r), w = h_2(u,r), \\ h_i(0,0) = 0, Dh_i(0,0) = 0, i = 1,2 \end{array} \right\}$$

for u and r sufficiently small.

Let

$$\delta \equiv u, \zeta \equiv (v,w), h = (h_1,h_2), Q = 0$$

and

$$R = \begin{bmatrix} -A & 0 \\ 0 & -\gamma-2\mu-\xi-\sigma \end{bmatrix}.$$

Assume that

$$\left. \begin{array}{l} h_1(u,r) = c_1u^2 + c_2ur + c_3r^2 + \cdots, \\ h_2(u,r) = c_4u^2 + c_5ur + c_6r^2 + \cdots \end{array} \right\}. \quad (7)$$

Using invariance of the graph of $h(u,r)$ under the dynamics generated by Equation (3), $h(u,r)$ must satisfy

$$
\begin{aligned}
\mathbb{N}(h(\delta,r)) &= D_\delta h(\delta,r)[Q\delta + f(\delta,h(\delta,r)r)] \\
&\quad -Rh(\delta,r) - g(\delta,h(\delta,r)r) \\
&= 0. \quad (8)
\end{aligned}
$$

Substitute $h = (h_1,h_2)$ from Equation (7) into Equation (8) and then compare the coefficients of u^2, ur and r^2, we obtain

$$c_1 = -\frac{m_2}{m_1}, c_2 = -\frac{m_6}{m_1}, c_3 = 0, c_4 = -\frac{n_2}{n_1}, c_5 = -\frac{n_6}{n_1}, c_6 = 0.$$

Hence

$$h_1(u,r) = -\frac{m_2}{m_1}u^2 - \frac{m_6}{m_1}ru, h_2(u,r) = -\frac{n_2}{n_1}u^2 - \frac{n_6}{n_1}ru.$$

Finally substituting the values of $v = h_1, w = h_2$ into Equations (5) and (6) we obtain the vector field reduced to the center manifold

$$
\begin{aligned}
\dot{u} &= r\,u\,l_5 + u^2\,l_1 + u^3\left(l_8 - \frac{l_3\,m_2}{m_1} - \frac{l_4\,n_2}{n_1}\right) \\
&\quad +ru^2\left(l_{14} - \frac{l_3\,m_6}{m_1} - \frac{l_4\,n_6}{n_1} - \frac{l_7\,n_2}{n_1}\right) + \cdots, \\
\dot{r} &= 0.
\end{aligned}
$$

Here we observe that $l_1 < 0$ and $l_5 > 0$. On the center manifold, we have

$$\frac{du}{dt} = G(u,r) = rul_5 + u^2 l_1,$$

with

$$
\begin{aligned}
G(0,0) &= G_u(0,0) = G_r(0,0),\\
G_{uu} &= 2l_1,\\
G_{ur} &= l_5,\\
G_{rr} &= 0.
\end{aligned}
$$

Here G_{ur} is positive and G_{uu} is negative. Hence, using transcritical bifurcation and center manifold theorems, worm-free equilibrium point is stable when $R_0 < 1$(since $r < 0$) and there is a separate unstable branch from the endemic equilibrium point and when $R_0 > 1$(since $r > 0$), worm-free equilibrium point becomes unstable while the separating branch becomes stable [9]. When $R_0 = 1$(since $r = 0$), center manifold is approximated by

$$\frac{du}{dt} \approx l_1 u^2 + \cdots.$$

Therefore, the worm-free equilibrium point is stable if it is approached from $u > 0$.

Hence, transcritical bifurcation occurs at the bifurcation point $R_0 = 1$. □

Theorem 2. *The endemic equilibrium point P^* is locally asymptotically stable if $\sigma\beta(S^*+c)S^* \le \eta_2(\gamma+\mu)(S^*+I^*+c)^2$ holds.*

Proof. The Jacobian matrix J^* at endemic equilibrium point P^* is given by

$$J^* = \begin{bmatrix} -\eta_1 - a_{21} & -\theta & -\theta + p\gamma - a_{23} \\ a_{21} & -\eta_2 & a_{23} \\ 0 & \sigma & -\mu-\gamma \end{bmatrix},$$

where

$$a_{21} = \frac{\beta(I^*+c)I^*}{(S^*+I^*+c)^2}, a_{23} = \frac{\beta(S^*+c)S^*}{(S^*+I^*+c)^2}.$$

The characteristic equation of J^* is

$$\lambda^3 + A_1\lambda^2 + A_2\lambda + A_3 = 0,$$

where

$$\begin{aligned} A_1 &= a_{21}+\gamma+\mu+\eta_1+\eta_2, \\ A_2 &= \theta(\mu+\eta_2)+\mu(\mu+2\eta_2)-a_{23}\sigma \\ &\quad +a_{21}(\gamma+\theta+\mu+\eta_2)+(\mu+\eta_2)\omega \\ &\quad +\gamma(\eta_1+\eta_2), \\ A_3 &= a_{21}(\gamma+\mu)(\theta+\mu+\xi)+a_{21}(q\gamma+\theta+\mu)\sigma \\ &\quad +\eta_1(-a_{23}\sigma+\eta_2(\gamma+\mu)), \end{aligned}$$

and

$$\begin{aligned} A_1A_2-A_3 &= a_{21}^2(\gamma+\theta+\mu+\eta_2) \\ &+(\gamma+\mu+\eta_2)(-a_{23}\sigma+(\gamma+\mu+\eta_1)(\eta_1+\eta_2)) \\ &+a_{21}(\gamma^2+\theta^2-a_{23}\sigma+\theta(2(2\mu+\eta_2)+\xi+\omega) \\ &+(\mu+\eta_2)(3\mu+\eta_2+2\omega)+2\gamma(\mu+\eta_1+\eta_2+\frac{p}{2}\sigma)). \end{aligned}$$

We observe that $A_1>0$ automatically satisfies. Both the conditions $A_3>0$ and $A_1A_2-A_3>0$ satisfies if $\sigma a_{23}\leq(\gamma+\mu)\eta_2$ implies that $\sigma\beta(S^*+c)S^*\leq\eta_2(\gamma+\mu)(S^*+I^*+c)^2$ holds. Hence, by Routh-Hurwitz criterion the endemic equilibrium point P^* is locally asymptotically stable. □

4.1 Global Stability Analysis

We analyze the global dynamics of worm-free and endemic equilibrium points. We find first the global stability of worm-free equilibrium point using the method developed in [3]. Rewrite the model system (2) as

$$\begin{cases} \frac{dY}{dt}=F(Y,Z), \\ \frac{dZ}{dt}=G(Y,Z),\ G(Y,0)=0. \end{cases} \tag{9}$$

where $Y=(S)$ and $Z=(E,I)$, with $Y\in\mathbb{R}$ denoting the number of susceptible node and $Z\in\mathbb{R}^2$ denoting the number of infected nodes (exposed and infectious). The worm-free equilibrium point is denoted by $Q_0=\left(Y^0,0\right)$. The following conditions $(A1)$ and $(A2)$ must give a local asymptotic stability:

(A1) Y^0 is globally asymptotic stable for $\frac{dY}{dt}=F(Y,0)$.

(A2) $G(Y,Z)=BZ-\hat{G}(Y,Z)$ where $\hat{G}(Y,Z)\geq0$ for $(Y,Z)\in U$,

and $B=D_zG\left(Y^0,0\right)$ is an M-matrix and U is the region of attraction. Then the following lemma holds.

Lemma 1. *The fixed point $Q_0=(Y^0,0)$ is a globally asymptotic stable equilibrium point of (9) if $R_0<1$ and Assumptions (A1) and (A2) are satisfied.*

Theorem 3. *Suppose $R_0<1$, then the worm-free equilibrium point P^0 is globally asymptotically stable.*

Proof. Let $Y=(S)$, $Z=(E,I)$ and $Q_0=(Y^0,0)$, where

$$Y^0=\frac{\Lambda}{\mu}\left(\frac{\mu+\theta}{\eta_1}\right). \tag{10}$$

Then,

$$\begin{aligned} \frac{dY}{dt} &= F(Y,Z) \\ &= \frac{\Lambda}{\mu}(\mu+\theta)-\eta_1S-\frac{\beta SI}{S+I+c}-\theta E-(\theta-(1-q)\gamma)I. \end{aligned}$$

At $S=S^0, F(Y,0)=0$, and

$$\frac{dY}{dt}=F(Y,0)=\frac{\Lambda}{\mu}(\mu+\theta)-\eta_1Y.$$

As $t\to\infty, Y\to Y^0$.

Hence $Y=Y^0(=S^0)$ is globally asymptotically stable.

Now,

$$\begin{aligned} G(Y,Z) &= \begin{bmatrix} -\eta_2 & \beta S^0 \\ \sigma & -(\mu+\gamma) \end{bmatrix}\begin{bmatrix} E \\ I \end{bmatrix} \\ &\quad -\begin{bmatrix} \beta S^0I-\frac{\beta SI}{S+I+c} \\ 0 \end{bmatrix}, \\ &= BZ-\hat{G}(Y,Z), \end{aligned}$$

where $B=\begin{bmatrix} -\eta_2 & \beta S^0 \\ \sigma & -(\mu+\gamma) \end{bmatrix}$ and $\hat{G}(Y,Z)=\begin{bmatrix} \beta S^0I-\frac{\beta SI}{S+I+c} \\ 0 \end{bmatrix}$.

In model system (2), total number of nodes is bounded by $N_1^0=\frac{\Lambda}{\mu}\frac{\mu+\theta}{\mu+\theta+\eta}$ where $\eta=\max\{\omega,\xi,q\gamma\}$, that is, $(S,E,I)\leq N_1^0$.

Since $S^0\geq N_1^0$, we have $S^0\geq N_1^0\geq S\geq\frac{S}{S+I+c}$ and thus, $\hat{G}(Y,Z)\geq0$. Therefore, B is an M-matrix. Hence $(A1)$ and $(A2)$ are satisfied and by Lemma 1, P^0 is globally asymptotically stable if $R_0<1$. □

Following Li and Muldowney [15], we obtain sufficient conditions for global asymptotic stability of the endemic equilibrium point. Consider the autonomous dynamical system:

$$\dot{x}=f(x)\ \text{ with } x(0,x_0)=x_0 \tag{11}$$

where $f:D\to\mathbb{R}^n, D\subset\mathbb{R}^n$ open set and simply connected and $f\in C^1(D)$. Let x^* be an equilibrium point of Equation (11) that is, $f(x^*)=0$. Assume that the following conditions hold:

(A3) There exists a compact absorbing set $K\subset D$.

(A4) Equation (11) has a unique equilibrium point x^* in D.

We know that if x^* is locally stable and all trajectories in D converges to x^* then it is to be globally stable in D. Bendixon criterion rule out the existence of non-constant periodic solutions of Equation (11) for $n\geq2$, that conditions satisfied by f. The classical Bendixson's condition $\text{div}f(x)<0$ for $n=2$, is robust under C^1 local perturbations of f.

Lemma 2. *Assume that conditions $(A3)$, $(A4)$ hold and Equation (11) satisfies a Bendixson criterion. Then, x^* is globally stable in D, provided it is stable.*

Let us consider $P(x)$ as $\binom{n}{2} \times \binom{n}{2}$ matrix-valued function that is C^1 on D and the matrix P_f has components

$$(p_{ij}(x))_f = \left(\frac{\partial p_{ij}(x)}{\partial x}\right)^T .f(x) = \nabla p_{ij}.f(x).$$

Assume that P^{-1} exists and is continuous for $x \in K$ (the compact absorbing set). The matrix $J^{[2]}$ is the second additive compound matrix of the Jacobian matrix J, that is $J(x) = Df(x)$. When $n = 3$, the second additive compound matrix of $J = (a_{ij})$ is given by [20],

$$J^{[2]} = \begin{bmatrix} a_{11} + a_{22} & a_{23} & -a_{13} \\ a_{32} & a_{11} + a_{33} & a_{12} \\ -a_{31} & a_{21} & a_{22} + a_{33} \end{bmatrix}.$$

Let $\mu(B)$ be the Lozinskii measure of B with respect to a vector norm $|.|$ in $\mathbb{R}^N.N = \binom{n}{2}$, defined by $\mu(B) = \lim_{h \to 0^+} \frac{|I+hB|-1}{h}$. A quantity $\bar{q}$ is defined as

$$\bar{q} = \limsup_{t \to \infty} \sup_{x \in K} \frac{1}{t} \int_0^t \mu(B(x(s, x_0)))\, ds,$$

where

$$B = P_f P^{-1} + P J^{[2]} P^{-1}.$$

If D is simply connected, $\bar{q} < 0$ rules out the presence of any orbit that gives rise to simple closed rectifiable curve that is invariant for Equation (11) and robust under C^1 local perturbations of f near any non-equilibrium point that is non-wandering. The following global stability result is given in Li and Muldowney [15].

Lemma 3. *Assume that D is simply connected and assumptions $(A3)$ and $(A4)$ hold. Then the unique equilibrium point x^* of Equation (11) is globally stable in D if $\bar{q} < 0$.*

Now, we analyze the global stability of the endemic equilibrium point x^*.

Theorem 4. *If $R_0 > 1, \xi + \sigma < \omega$ and $(1-q)\gamma \geq \theta$, then the endemic equilibrium P^* of the model system (2) is globally stable in U.*

Proof. From Theorem 2, if the endemic equilibrium point P^* exists, is locally asymptotically stable. From Theorem 1, when $R_0 > 1$, P^0 is unstable. The instability of P^0 together with $P^0 \in \partial U$ is implies to the uniform persistence [6], that is there exists a constant $C > 0$ such that:

$$\lim_{t \to \infty} \inf x(t) > C, x = (S, E, I).$$

The uniform persistence together with the boundedness of U, is equivalent to the existence of a compact set in the interior of U which is absorbing for the model system (2) [10]. Thus, $(A3)$ is verified. Now, the second additive compound matrix $J^{[2]}(S, E, I)$ is given by

$$J^{[2]} = \begin{bmatrix} \Phi_1 & \frac{\beta(S+c)S}{(S+I+c)^2} & \frac{\beta(S+c)S}{(S+I+c)^2} + \theta - (1-q)\gamma \\ \sigma & \Phi_2 & -\theta \\ 0 & \frac{\beta(I+c)I}{(S+I+c)^2} & -(\mu + \gamma + \eta_2) \end{bmatrix}$$

where

$$\begin{aligned} \Phi_1 &= -(\eta_1 + \eta_2) - \frac{\beta(I+c)I}{(S+I+c)^2}, \\ \Phi_2 &= -(\mu + \gamma + \eta_1) - \frac{\beta(I+c)I}{(S+I+c)^2}. \end{aligned}$$

Let us consider

$$P = P(S, E, I) = diag\left\{1, \frac{E}{I}, \frac{E}{I}\right\}.$$

Therefore,

$$P_f P^{-1} = diag\left\{0, \frac{\dot{E}}{E} - \frac{\dot{I}}{I}, \frac{\dot{E}}{E} - \frac{\dot{I}}{I}\right\}.$$

Then $B = P_f P^{-1} + P J^{[2]} P^{-1}$

$$= \begin{bmatrix} \Phi_1 & \frac{\beta(S+c)S}{(S+I+c)^2}\frac{I}{E} & \left(\frac{\beta(S+c)S}{(S+I+c)^2} + \theta - (1-q)\gamma\right)\frac{I}{E} \\ \sigma\frac{E}{I} & \frac{\dot{E}}{E} - \frac{\dot{I}}{I} + \Phi_2 & -\theta \\ 0 & \frac{\beta(I+c)I}{(S+I+c)^2} & \frac{\dot{E}}{E} - \frac{\dot{I}}{I} - (\mu + \gamma + \eta_2) \end{bmatrix}.$$

Let

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix},$$

where

$$\begin{aligned} B_{11} &= \Phi_1 = -\left(\eta_1 + \eta_2 + \tfrac{\beta(I+c)I}{(S+I+c)^2}\right), \\ B_{12} &= \begin{bmatrix} \frac{\beta(S+c)S}{(S+I+c)^2}\frac{I}{E} & \left(\frac{\beta(S+c)S}{(S+I+c)^2} + \theta - (1-q)\gamma\right)\frac{I}{E} \end{bmatrix}, \end{aligned}$$

$$B_{21} = \begin{bmatrix} \sigma\frac{E}{I} \\ 0 \end{bmatrix},$$

$$B_{22} = \begin{bmatrix} \frac{\dot{E}}{E} - \frac{\dot{I}}{I} - \frac{\beta(I+c)I}{(S+I+c)^2} - (\mu + \gamma + \eta_1) & -\theta \\ \frac{\beta(I+c)I}{(S+I+c)^2} & \frac{\dot{E}}{E} - \frac{\dot{I}}{I} - (\mu + \gamma + \eta_2) \end{bmatrix}.$$

Now, consider the norm $|(u_1, u_2, u_3)| = \max\{|u_1|\, |u_2| + |u_3|\}$ in $\mathbb{R}^3$, where (u_1, u_2, u_3) denotes vector in $\mathbb{R}^3$ and the Lozinskii measure is denoted by μ with respect to this norm [17].

$$\begin{aligned} \mu(B) &\leq \sup\{g_1, g_2\}, \\ &= \sup\left\{\mu_1(B_{11}) + |(B_{12})|, \mu_1(B_{22}) + |(B_{21})|\right\}. \end{aligned}$$

where $|B_{21}|\, |B_{12}|$ are matrix norms with respect to the L^1 vector norm, and μ_1 denotes the Lozinskii measure with respect to the L^1 norm.

Then

$$\begin{aligned} g_1 &= \mu_1(B_{11}) + |B_{12}|, \\ &= -\left(\eta_1 + \eta_2 + \frac{\beta(I+c)I}{(S+I+c)^2}\right) + \frac{\beta(S+c)S}{(S+I+c)^2}\frac{I}{E} \\ &\quad + \frac{I}{E}\max\{0, \theta - (1-q)\gamma\}, \\ &\leq -\left(\eta_1 + \eta_2 + \frac{\beta(I+c)I}{(S+I+c)^2}\right) + \frac{\beta(S+c)S}{(S+I+c)^2}\frac{I}{E}, \\ &\qquad\qquad [\text{when } (1-q)\gamma \geq \theta] \\ &\leq -\eta_1 + \frac{\dot{E}}{E} - \left[\frac{\beta S}{S+I+c} - \frac{\beta(S+c)S}{(S+I+c)^2}\right]\frac{I}{E}, \end{aligned}$$

[from steady state of second equation of model system (2)]

$$\begin{aligned} &= \frac{\dot{E}}{E} - \eta_1 - \frac{\beta SI}{(S+I+c)^2}\frac{I}{E}, \\ &\leq \frac{\dot{E}}{E} - \mu. \end{aligned}$$

Again,

$$\begin{aligned} g_2 &= |B_{21}| + \mu_1(B_{22}), \\ &= \sigma\frac{E}{I} + \frac{\dot{E}}{E} - \frac{\dot{I}}{I} - 2\mu - \gamma - \theta \\ &\quad + \max\{-\omega, -(\sigma+\xi)\}, \\ &= \frac{\dot{E}}{E} - \mu - \theta + \max\{-\omega, -(\sigma+\xi)\}, \end{aligned}$$

[from steady state of third equation of model system (2)]

$$\begin{aligned} &\leq \frac{\dot{E}}{E} - \mu - \theta, \quad [\text{when } \xi + \sigma < \omega] \\ &\leq \frac{\dot{E}}{E} - \mu. \end{aligned}$$

Therefore,

$$\mu(B) \leq \sup\{g_1, g_2\} = \frac{\dot{E}}{E} - \mu.$$

Along each solution $(S(t), E(t), I(t))$ of the system with $(S(0), E(0), I(0)) \in K$, where K is the compact absorbing set, we have

$$\frac{1}{t}\int_0^t \mu(B)\, ds \leq \frac{1}{t}\log\frac{E(t)}{E(0)} - \mu,$$

which implies that

$$\bar{q} = \limsup_{t \to \infty} \sup_{x_0 \in U} \frac{1}{t}\int_0^t \mu(B(x(s, x_0)))\, ds \leq -\mu < 0.$$

Therefore, $\bar{q} < 0$ and thus Bendixson criteria is also fulfilled. Hence the global stability of the endemic or worm-induced equilibrium point has established. □

5 Numerical Simulations

Numerically, Runge-Kutta method is used to simulate the model system (2) using MATLAB software. The dynamical behaviors of all the three classes S, E and I are observed by considering a set of parameter values and initial conditions. We have taken initial condition $(22, 20, 20)$ and parametric values

$$\begin{aligned} &\Lambda = 0.4, \mu = \frac{1}{(65 * 365)}, \theta = \frac{1}{(2 * 365)}, \gamma = \frac{1}{30}, \\ &\omega = 0.6, \beta = 0.14, c = 10, q = 0.9, \xi = 0.2, \sigma = 0.1. \end{aligned} \tag{12}$$

The endemic equilibrium point for the vaccination rate $\omega = 0.1$ is $(102.2687, 10.2488, 30.7065)$ and the worm-free equilibrium point for vaccination rate $\omega = 0.9$ is $(14.8677, 0, 0)$ and other parameter values are same as used in (12). The analysis of Figures 2 and 3 under different vaccination rate shows the stability of both worm-free and endemic equilibrium points that is, for the cases when $R_0 <$ or > 1. We have critically examined the infectious class I for the different values of $\xi(= \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7})$ which support the reality that as the self-prevention techniques of nodes ξ decreases, infection increases (Figure 4) and the increment in transmission rate $\beta(= 0.8, 0.12, 0.16, 0.20, 0.24)$ causes increment in infection (Figure 5) that is infected node increases and susceptible node decreases (Figure 6) and will be stable. It is observed from Figure 7, the values of transmission rate β decreases the susceptible nodes attain its saturation value, when $R_0 < 1$. We have also observed the evolutions of susceptible nodes for the fraction of recovered nodes gaining worm-acquired immunity, q (Figure 8) and observation tells that the susceptible node increases when the fraction of recovered nodes gaining worm-acquired immunity, $q(= 0.2, 0.4, 0.6, 0.8, 1.0)$ increases when $\beta = 0.12$ $(R_0 < 1)$ but when $\beta = 0.20$ $(R_0 > 1)$ then the susceptible nodes attain its saturation value. The above parametric values given in (12) satisfies our analytical results.

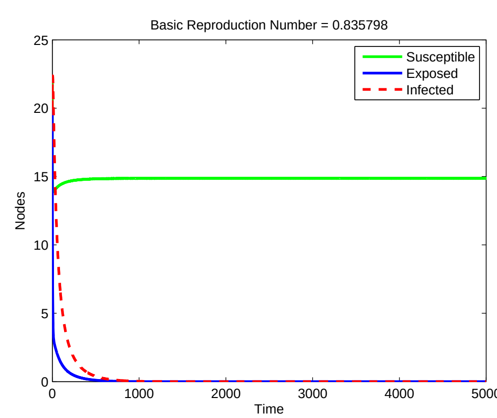

Figure 2: Time series of susceptible, exposed and infected nodes when $\omega = 0.9$

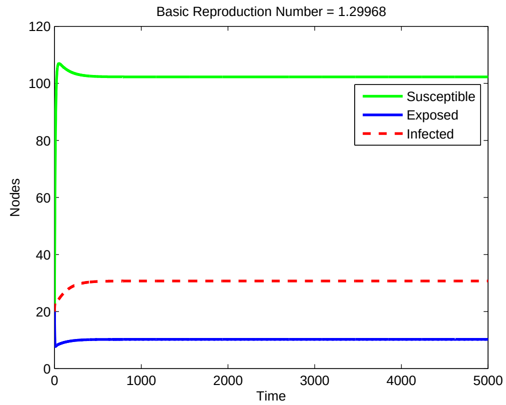

Figure 3: Time series of susceptible, exposed and infected nodes when $\omega = 0.1$

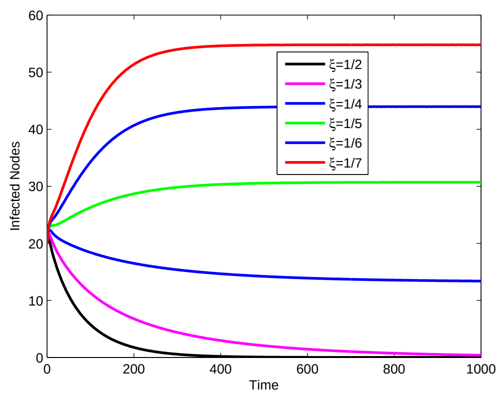

Figure 4: Dynamical behavior of infected class for different values of ξ when $\omega = 0.1$

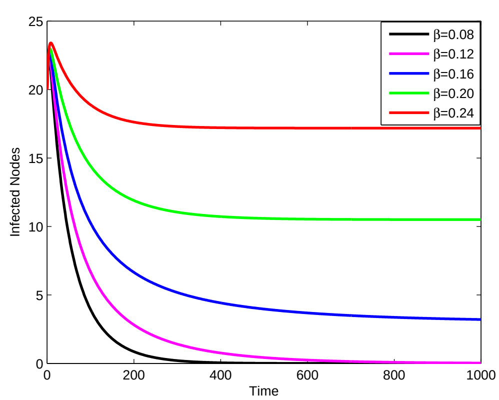

Figure 5: Dynamical behavior of infected class for different values of β

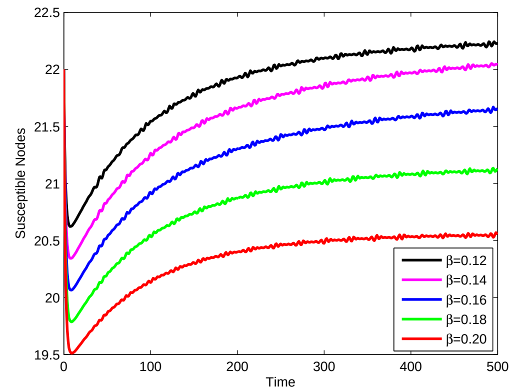

Figure 6: Dynamical behavior of susceptible class for different values of β when $R_0 > 1$

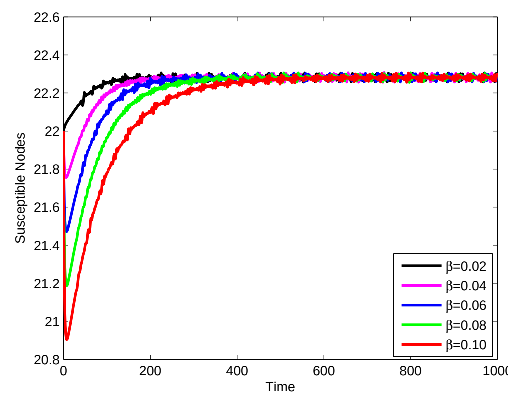

Figure 7: Dynamical behavior of susceptible class for different values of β when $R_0 < 1$

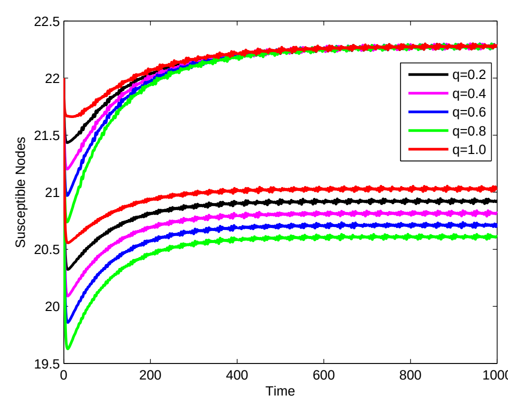

Figure 8: Dynamical behavior of susceptible nodes for different values of q when $\beta = 0.12$ ($R_0 < 1$) and $\beta = 0.20$ ($R_0 > 1$)

6 Discussions and Conclusions

An e-epidemic $SVEIS$ model with nonlinear incidence rate has been proposed for the transmission of worms in the computer network. Stability analysis and behavior of the reduced model system (2) have been investigated for both worm-free and endemic equilibrium points. Local stability analysis is established by using Routh-Hurwitz criterion. Characteristics of basic reproduction number have been discussed and found that if $R_0 < 1$, WFE point P^0 is globally asymptotically stable under certain conditions and worm declines from the computer network, where as if $R_0 > 1$, the worm-free equilibrium point is unstable and worm persists. In Figure 9, the forward transcritical bifurcation occurs at $R_0 = 1$ and it is effectively eradicate the worms. When the bifurcation parameter R_0, crosses the bifurcation threshold $R_0 = 1$, the endemic

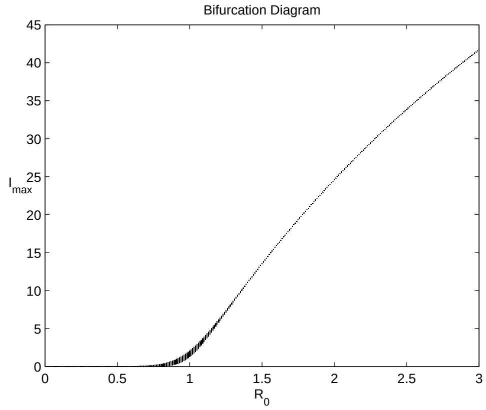

Figure 9: Transcritical Bifurcation diagram in the plane $(R_0, I_{\max})$

equilibrium point enters into the positive orthant. The vaccination rate reaches its critical value

$$\omega_c = \frac{(\mu+\theta)}{\mu c}\left(\frac{\sigma\beta\Lambda}{(\mu+\gamma)(\mu+\xi+\sigma)} - (\Lambda+\mu c)\right)$$

then the basic reproduction number $R_0 = 1$. We have observed the effect of the vaccination rate ω, on the basic reproduction number which ultimately affecting the dynamics of the model system. The optimal vaccine at critical level is most important factor to effectively eradicate the worm. Hence vaccination rate, ω must be greater than critical vaccination rate, $\omega_c = 0.531956$ to control worm from the network otherwise, worm persists in the network.

To study the affect of the parameter q, a fraction of recovered nodes on the dynamics of the model systems we find that it does not appear in the definition of R_0 and w_c. Due to waning of vaccination, a fraction of recovered nodes $(1-q)\gamma I$ moves to the susceptible class directly and rest via vaccinated class [23]. The effect of q on the susceptible node for transmission rate $\beta = 0.12$ and 0.20 is shown in Figure 8. In self-replicating computer worms modeling, the nonlinear incidence rate plays a major role and ensures that the model system can give a reasonable qualitative description of the worm dynamics.

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